Problem of the Week
Problem D and Solution
This is the Year

Problem

The positive integers can be arranged as follows.

Row 1	1					
Row 2	2	3				
Row 3	4	5	6			
Row 4	7	8	9	10		
Row 5	11	12	13	14	15	
Row 6	16	17	18	19	20	21
⋮						

More rows and columns continue to list the positive integers in order, with each new row containing one more integer than the previous row. How many integers less than 2020 are in the *column* that contains the number 2020?

Solution

In the table given, there is one number in Row 1, there are two numbers in Row 2, three numbers in Row 3, and so on, with n numbers in Row n.
The numbers in the rows list the positive integers in order beginning at 1 in Row 1, with each new row containing one more integer than the previous row. Thus, the last number in each row is equal to the sum of the number of numbers in each row of the table up to that row.

For example, the last number in Row 4 is 10, which is equal to the sum of the number of numbers in rows 1, 2, 3, and 4. But the number of numbers in each row is equal to the row number. So 10 is equal to the sum $1 + 2 + 3 + 4$.
That is, the last number in Row n is equal to the sum

$$1 + 2 + 3 + 4 + \ldots + (n - 1) + n = \frac{n(n+1)}{2}.$$

To find out which row the integer 2020 occurs in, we could use organized trial and error.
Using trial and error, we find that since $\frac{63(63+1)}{2} = 2016$, then the last number in Row 63 is 2016.
We then find $\frac{64(64+1)}{2} = 2080$, so the last number in Row 64 is 2080.
Since 2020 is between 2016 and 2080, then it must appear somewhere in the 64th row.

Alternatively, we could determine the row the integer 2020 occurs in by using the quadratic formula to solve the equation $\frac{n(n+1)}{2} = 2020$ for n.

$$\begin{aligned} \frac{n(n+1)}{2} &= 2020 \\ n(n+1) &= 4040 \\ n^2 + n - 4040 &= 0 \\ n &\approx 63.1 \text{ (using the quadratic formula and } n \geq 0) \end{aligned}$$

This means that the integer 2020 will be in Row 64.

If we look at the original arrangement given, the first number in Row 6, which is 16, has five numbers in the column above it. The second number in Row 6, which is 17, has four numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.
Also, note that the first number in Row 5, which is 11, has four numbers in the column above it. The second number in Row 5, which is 12, has three numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.
The pattern is that the first number in Row n has $n-1$ numbers in the column above it. The second number in Row n has $n-2$ numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.

Therefore, the first number of Row 64, which is 2017, will have 63 numbers in the column above it. The second number of Row 64, which is 2018, will have 62 numbers in the column above it. The third number of Row 64, which is 2019, will have 61 numbers in the column above it. The fourth number of Row 64, which is 2020, will have 60 numbers in the column above it.

Therefore, there are 60 integers less than 2020 in the column that contains the number 2020.